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Equational Languages

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This paper deals with equations whose solutions are vectors of languages. Formally, solutions of equations are fix points of vectorial functions on languages. On the other hand equations (and sets of equations) can be considered as grammars. Three main groups of problems are dealt with: (1) solvability of equations in a lattice of languages, (2) relationship between type of functions used in equations and properties of languages defined by them, (3) applications to the theory of context-free and regular languages.

1. INTRODUCTION

Two principal language-describing tools known in the theory of formal languages are generative (Chomsky's) grammars and accepting automata. On the other hand, in describing programming languages one uses mostly so-called Backus–Naur equations. These, although called equations, are not treated in a proper algebraic sense, but rather as formal expressions "to be referred to as equations." In fact, $::=$ stands for $=$ and $|$ stands for $\cup$ (the union), but this is never improved in practice. Moreover, to describe the meaning of such equations one associates with them context-free grammars. In effect the "equational" character of equations is lost.

Sets of equations describing languages have previously been investigated by other authors. Chomsky and Schutzenberger (1963) discussed such equations understood as formal expressions. This idea was later developed in Mezei and Wright (1967) and in Shamir (1967). Equations have been treated as formal expressions which offers considerable difficulty in formulating and proving theorems. In the meantime Ginsburg and Rice (1962) and Ginsburg (1966) described Backus–Naur equations by associating with them sets of certain many-argumental functions on languages and used this tool in proving the equivalence of ALGOL-like and context-free concepts. The ideas of Ginsburg and Rice are generalized and developed in this paper. An earlier exposition of the present results was given in a technical report (Blikle, 1971).

2. Basic Notions

The reader is assumed to be familiar with basic notions of the theory of formal languages like *word*, *language*, *concatenation*, *substitution*, *homomorphism*, etc. [cf. Ginsburg (1966)].

Let V be an arbitrary (finite) alphabet to be fixed for the sequel. For any positive integer n the *n-dimensional lattice of languages* is the set

$$\mathscr{L}^n = (2^{V^*})^n$$

of all vectors $(A_1, \ldots, A_n)$ with $A_i \subseteq V^*$ for $i = 1, \ldots, n$. Clearly $\mathscr{L}^n$ is a lattice (complete) with respect to the ordering $\subseteq^n$ defined by the formula

$$(A_1, \ldots, A_n) \subseteq^n (B_1, \ldots, B_n) \underset{\text{def}}{\equiv} (\forall i \leqslant n)(A_i \subseteq B_i).$$

In the sequel we shall omit the superscripts and write $\subseteq$ instead of $\subseteq^n$. The lattice operations $\cup$, $\cap$, $\bigcup$ and $\bigcap$ in $\mathscr{L}^n$ and the n-dimensional concatenation will be also written without superscripts, e.g.,

$$(A_1, \ldots, A_n) \circ (B_1, \ldots, B_n) \underset{\text{def}}{=} (A_1 \circ B_1, \ldots, A_n \circ B_n),$$

etc.

Elements of $\mathscr{L}^n$ will usually be denoted by boldface latin initial capitals $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$. Boldface latin terminals $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$ will denote variables ranging over $\mathscr{L}^n$. Moreover, $\boldsymbol{\varnothing}^n = (\underbrace{\varnothing, \ldots, \varnothing}_{n})$, where $\varnothing$ is the empty set.

In the paper we shall be concerned with total functions of the type $F : \mathscr{L}^n \to \mathscr{L}^m$. The cases of particular interest are $n = m$—*vectorial functions* (abr. v *functions*), and $m = 1$—*scalar functions* (abr. s *functions*).

A function $F : \mathscr{L}^n \to \mathscr{L}^m$ is called *monotonic* if $\mathbf{A} \subseteq \mathbf{B}$ implies $F(\mathbf{A}) \subseteq F(\mathbf{B})$ for any $\mathbf{A}$ and $\mathbf{B}$ in $\mathscr{L}^n$.

F is said to be *continuous*, if $F(\bigcup_{i=1}^{\infty} \mathbf{A}_i) = \bigcup_{i=1}^{\infty} F(\mathbf{A}_i)$ for any increasing sequence $\mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \cdots$ of vectors in $\mathscr{L}^n$. It can be interesting to note that this notion of continuity coincides with the one of D. Scott [cf. Scott (1970)] for the lattice of languages $\mathscr{L}^n$. However it seems to be different for arbitrary complete lattices.

As it is easy to see, each continuous function is monotonic. Examples of continuous functions are "polynomials" constructed with the operations of union, concatenation and $*$ closure, e.g., $F(X, Y) = (AX \cup Y^*, AXBY^*)$ is a 2-dimensional continuous function and $G(X, Y) = (X - Y, Y - X)$ is neither continuous nor monotonic.

Consider an arbitrary vectorial function $F : \mathscr{L}^n \to \mathscr{L}^n$. If there exists a vector $\mathbf{A}$ in $\mathscr{L}^n$ with $F(\mathbf{A}) = \mathbf{A}$, then $\mathbf{A}$ is said to be a *fix point* (abr. FP) of F. If $\mathbf{A}$ is an FP of F and for any other FP $\mathbf{B}$ of F, $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{A}$ is said to be *the least fix point* (abr. LFP) of F. We shall denote it by $\|F\|$.

The following approximation theorem is well known [cf. Mezei and Wright (1967), Shamir (1967), Scott (1970) and others].

THEOREM 2.1. *For every continuous function $F : \mathscr{L}^n \to \mathscr{L}^n$ the* LFP $\|F\|$ *exists and*

$$\|F\| = \bigcup_{i=1}^{\infty} F^i(\varnothing^n),$$

where $F^i(X) = F \cdots F(X)$ i times.

Let $F(\mathbf{X}, \mathbf{Y})$ be an arbitrary $n + m$-argumental function, where $\mathbf{X}$ ranges over $\mathscr{L}^n$, $\mathbf{Y}$ ranges over $\mathscr{L}^m$ and F ranges over $\mathscr{L}^n$ again. In other words $F : \mathscr{L}^{n+m} \to \mathscr{L}^n$. If there exists a function $G : \mathscr{L}^m \to \mathscr{L}^n$ such that for every vector $\mathbf{A}$ in $\mathscr{L}^m$, $G(\mathbf{A})$ is the LFP of $F(\mathbf{X}, \mathbf{A})$, then G is said to be the *resolvent of F with respect to* $\mathbf{X}$ and is written [cf. De Bakker (1971)]

$$G(\mathbf{Y}) = (\mu\mathbf{X}) F(\mathbf{X}, \mathbf{Y}).$$

As it is easy to see, $G(\mathbf{Y})$ is a solution of the equation

$$\mathbf{X} = F(\mathbf{X}, \mathbf{Y}),$$

therefrom the name. On the other hand if $m = 0$, then $(\mu\mathbf{X}) F(\mathbf{X}) = \|F\|$ provided $\|F\|$ exists.

The following theorem is proved in Leszczyłowski (1971):

THEOREM 2.2. *For every continuous function $F(\mathbf{X}, \mathbf{Y})$, where $\mathbf{X}$ and F ranges over $\mathscr{L}^n$, the resolvent $G(\mathbf{Y}) = (\mu\mathbf{X}) F(\mathbf{X}, \mathbf{Y})$ exists and is a continuous function.*

With the help of Theorem 2.1 one can easily prove the following formulas for arbitrary languages A, B and C:

$$\begin{aligned} &(\mu X)X = \varnothing, && (\mu X)(XA \cup B) = BA^*, \\ &(\mu X)A = A, && (\mu X)(X^*A \cup A) = A^+, \\ &(\mu X)(AX \cup B) = A^*B, && (\mu X)(AXB \cup C) = \bigcup_{i=0}^{\infty} A^iCB^i, \end{aligned} \tag{1}$$

etc.

3. Sets of Equations

Let $\mathbf{A} \in \mathscr{L}^n$. By $[\mathbf{A}]_i$ for $i \leqslant n$ we shall denote the *i-th coordinate of* $\mathbf{A}$, i.e., if $\mathbf{A} = (A_1, ..., A_n)$, then $[\mathbf{A}]_i = A_i$.

With every function $F : \mathscr{L}^n \to \mathscr{L}^m$ we can associate a vector $(f_1, ..., f_m)$ of n-argumental scalar functions defined as follows: for every $\mathbf{A}$ in $\mathscr{L}^n$ and every $i \leqslant m, f_i(\mathbf{A}) = [F(\mathbf{A})]_i$. For the sake of convenience we shall frequently identify the function F with the corresponding vector of functions $(f_1, ..., f_m)$ and write simply $F = (f_1, ..., f_m)$. Clearly F is continuous if $f_1, ..., f_m$ are all continuous.

Consider now a set of n-argumental scalar functions $f_1, ..., f_n$ and the following set of equations:

$$\begin{aligned} X_1 &= f_1(X_1, ..., X_n) \\ &\cdots \\ X_n &= f_n(X_1, ..., X_n). \end{aligned} \tag{2}$$

Any fix point of the function $F = (f_1, ..., f_n)$ is said to be a *solution* of this set. If $\| F \|$ exists, then $\| F \|$ is said to be *the least solution* of (2).

To simplify the notation (2) will be frequently written as

$$\mathbf{X} = F(\mathbf{X}).$$

Similarly, if $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_m)$, then by $(\mathbf{A}, \mathbf{B})$ we shall mean the vector $(A_1, ..., A_n, B_1, ..., B_m)$.

By a *set of equations with parameters* we shall mean any set of equations of the form

$$\mathbf{X} = F(\mathbf{X}, \mathbf{Y}),$$

where $\mathbf{X}$ and F range over the same $\mathscr{L}^n$. Now if $G(\mathbf{Y}) = (\mu\mathbf{X}) F(\mathbf{X}, \mathbf{Y})$ exists, then it is called *the solution* of this set. Clearly, $F(G(\mathbf{Y}), \mathbf{Y}) = G(\mathbf{Y})$ for any $\mathbf{Y}$.

Two sets of equations $\mathbf{X} = F(\mathbf{X})$ and $\mathbf{Y} = H(\mathbf{Y})$ are said to be *equivalent* if either both $\| F \|$ and $\| G \|$ do not exist or both exist and are equal.

Theorem 3.1. *Consider the set of equations of the form*

$$\mathbf{X} = F(\mathbf{X}, \mathbf{Y}),$$

$$\mathbf{Y} = H(\mathbf{X}, \mathbf{Y}),$$

where $F : \mathscr{L}^{n+m} \to \mathscr{L}^n$, $H : \mathscr{L}^{n+m} \to \mathscr{L}^m$, $\mathbf{X}$ *ranges over* $\mathscr{L}^n$ *and* $\mathbf{Y}$ *ranges over* $\mathscr{L}^m$.

If F and H are continuous functions and $G(\mathbf{Y}) = (\mu\mathbf{X})F(\mathbf{X}, \mathbf{Y})$, then the set of equations

$$\mathbf{X} = G(\mathbf{Y}),$$
$$\mathbf{Y} = H(G(\mathbf{Y}), \mathbf{Y}),$$

is equivalent with the set (3).

This theorem is proved in Leszczyłowski (1971) and permits resolving equations in $\mathscr{L}^n$ by eliminating variables. It is to be emphasized that the theorem is not true for arbitrary F and G, i.e., where F and G are not continuous.

4. Equational Languages

Let $\mathscr{F}$ be an arbitrary family of scalar functions. A language A is said to be *equational with respect to* $\mathscr{F}$ if there exists a v-function $F = (f_1, ..., f_n)$ with $f_1, ..., f_n$ in F, such that $\|F\|$ exists and $A = [\|F\|]_i$ for some $i \leqslant n$. By $\mathrm{EQ}[\mathscr{F}]$ we denote the set of all languages equational with respect to $\mathscr{F}$.

Theorem 4.1. *Let $\mathscr{F}$ be an arbitrary family of continuous functions. The family of languages $\mathrm{EQ}[\mathscr{F}]$ is closed under all functions in $\mathscr{F}$ and under the resolvents of all functions in $\mathscr{F}$.*

Proof. Let $g(X_1, ..., X_n)$ be in $\mathscr{F}$ and let $A_1, ..., A_n$ be in $\mathrm{EQ}[\mathscr{F}]$. By virtue of the assumption there exists a v-function F over $\mathscr{F}$ with $A_i = [\|F\|]_i$ for $i = 1, ..., n$. Consider now the set of equations

$$\mathbf{X} = F(\mathbf{X}),$$
$$Y = g(\mathbf{X}, Y),$$

and let $(A_1, ..., A_n) = \mathbf{A}$. By Theorem 3.1, $(\mathbf{A}, (\mu Y)g(\mathbf{A}, Y))$ is the least solution of this set. Therefore $(\mu Y)g(\mathbf{A}, Y) \in \mathrm{EQ}[\mathscr{F}]$, which proves the second part of the assertion. The first part follows immediately therefrom since every function in $\mathscr{F}$ is a resolvent of a function in $\mathscr{F}$. Indeed, $f(\mathbf{X}) = (\mu Y)f(\mathbf{X})$. Q.E.D.

Consider now an arbitrary family $\mathscr{F}$ of continuous functions and let $\mathrm{RCC}[\mathscr{F}]$ (to be read: resolvent-composition closure) denotes the least set of functions that contains $\mathscr{F}$ as a subset and is closed under the μ operation and the composition of functions.

THEOREM 4.2. *If $\mathscr{F}$ is a family of continuous functions and contains the identity function $f(X) = X$, then*

$$\mathrm{EQ}[\mathscr{F}] = \mathrm{EQ}[\mathrm{RCC}[\mathscr{F}]].$$

Proof. Let $\mathscr{F}$ be an arbitrary family of continuous functions with $f(X) = X$. Consider an increasing sequence of sets of functions $\mathscr{F}_0, \mathscr{F}_1, \ldots$ defined as follows: (1) $\mathscr{F}_0 = \mathscr{F}$, (2) $\mathscr{F}_{i+1}$ is the least set with the following properties:

(i) $\mathscr{F}_i \subseteq \mathscr{F}_{i+1}$;

(ii) if F, G are in $\mathscr{F}_i$, then the resolvent of F with respect to any vector of variables, and any composition of F and G are in $\mathscr{F}_{i+1}$.

Clearly $\mathrm{RCC}[\mathscr{F}] = \bigcup_{i=0}^{\infty} \mathscr{F}_i$.

Consider now the set of equations

$$\begin{aligned} \mathbf{X} &= (\mu\mathbf{Z})\, F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}), \\ \mathbf{Y} &= H(\mathbf{X}, \mathbf{Y}), \end{aligned} \tag{4}$$

and the set

$$\begin{aligned} \mathbf{Z} &= F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}), \\ \mathbf{X} &= \mathbf{Z}, \\ \mathbf{Y} &= H(\mathbf{X}, \mathbf{Y}). \end{aligned} \tag{5}$$

By Theorem 3.1 the last set is equivalent with

$$\begin{aligned} \mathbf{Z} &= (\mu\mathbf{Z})\, F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \\ \mathbf{X} &= (\mu\mathbf{Z})\, F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \\ \mathbf{Y} &= H(\mathbf{X}, \mathbf{Y}). \end{aligned} \tag{6}$$

Clearly, if $(\mathbf{A}, \mathbf{B})$ is the least solution of (4), then $(\mathbf{A}, \mathbf{A}, \mathbf{B})$ is the least solution of (6), hence also of (5).

In a similar way we can show that the set of equations

$$\begin{aligned} \mathbf{X} &= F(\mathbf{X}, G(\mathbf{X}, \mathbf{Y}), \mathbf{Y}), \\ \mathbf{Y} &= H(\mathbf{X}, \mathbf{Y}), \end{aligned}$$

can be "replaced" by the set

$$\begin{aligned} \mathbf{Z} &= G(\mathbf{X}, \mathbf{Y}), \\ \mathbf{X} &= F(\mathbf{X}, \mathbf{G}(\mathbf{X}, \mathbf{Y}), \mathbf{Y}), \\ \mathbf{Y} &= H(\mathbf{X}, \mathbf{Y}). \end{aligned}$$

In effect we have proved that every set of equations in $\mathscr{F}_{i+1}$ can be replaced (in a sense) by a corresponding set in $\mathscr{F}_i$. In other words $\mathrm{EQ}[\mathscr{F}_{i+1}] \subseteq \mathrm{EQ}[\mathscr{F}_i]$, which completes the proof. Q.E.D.

This theorem together with Theorem 4.1 imply immediately what follows:

COROLLARY 4.1. *Let $\mathscr{F}$ be an arbitrary family of continuous functions containing the identity function $f(X) = X$. The equational family of languages* $\mathrm{EQ}[\mathscr{F}]$ *is closed under all functions in* $\mathrm{RCC}[\mathscr{F}]$.

It should be stressed that $\mathrm{RCC}[\mathscr{F}]$ need not contain all functions that do not lead out of $\mathrm{EQ}[\mathscr{F}]$. For appropriate examples see Blikle (1971).

By an *operation on languages* we shall mean any function $f : \mathscr{L}^1 \to \mathscr{L}^1$. Particular operations on languages are substitutions, homomorphisms and inverse homomorphisms. As it is easy to show [see Blikle (1971)] all these operations are continuous.

Given an operation $h : \mathscr{L}^1 \to \mathscr{L}^1$ we shall denote

$$h^n(X_1, \dots, X_n) \underset{\mathrm{def}}{=} (h(X_1), \dots, h(X_n)).$$

Given two n-argumental functions F and G, $F \circ G$ will denote the composition, i.e., $[F \circ G](X) = G(F(X))$.

THEOREM 4.3. *Let $\mathscr{F}$ be an arbitrary family of continuous functions and let h be an arbitrary continuous operation with $h(\varnothing) = \varnothing$. If for every n-argumental* v*-function F in $\mathscr{F}$ there exists a* v*-function F_1 in $\mathscr{F}$ with*

$$F \circ h^n = h^n \circ F_1,$$

then the family of equational languages $\mathrm{EQ}[\mathscr{F}]$ *is closed under h, i.e.,* $h(\mathrm{EQ}[\mathscr{F}]) \subseteq \mathrm{EQ}[\mathscr{F}]$.

Proof. Let h be an arbitrary continuous operation with $h(\varnothing) = \varnothing$ and for some F in $\mathscr{F}$ let there exist F_1 in $\mathscr{F}$ with $F \circ h^n = h^n \circ F_1$. Consider now two sets of equations

$$X = F(X) \qquad \text{and} \qquad X = F_1(X)$$

and let $\mathbf{A}$ be the least solution of the first one, and $\mathbf{B}$ of the second one. We shall show the equality

$$\mathbf{B} = h^n(\mathbf{A}).$$

Indeed, By Theorem 2.1,

$$B = \bigcup_{i=1}^{\infty} F_1^i(\varnothing^n). \tag{7}$$

On the other hand,

$$F_1(\varnothing^n) = F_1(h^n(\varnothing^n)) = h^n(F(\varnothing^n)).$$

Therefore, by an inductive argument,

$$F_1^i(\varnothing^n) = h^n(F^i(\varnothing^n))$$

for $i = 1, 2, \ldots$. Hence, by (7),

$$\mathbf{B} = \bigcup_{i=1}^{\infty} h^n(F^i(\varnothing^n)) = h\left(\bigcup_{i=1}^{\infty} F^i(\varnothing^n)\right) = h^n(\mathbf{A}). \qquad \text{Q.E.D.}$$

5. Inductive Families of Functions

Consider an arbitrary finite set $e_1, \ldots, e_n$ of scalar functions and an arbitrary family $\mathscr{L}_0$ of languages. By the *inductive family of functions* over $e_1, \ldots, e_n$ and $\mathscr{L}_0$, in symbols

$$\mathrm{IND}[e_1, \ldots, e_n, \mathscr{L}_0],$$

we mean the least set of many-argumental scalar functions which contains

(1°) all projections $f(X_1, \ldots, X_n) = X_i$ with $i \leqslant n$,

(2°) all constant functions with values in $\mathscr{L}_0$,

(3°) the functions $e_1, \ldots, e_n$,

and which is closed under the operation of composition.

Consider now an infinite sequence $\mathscr{F}_0, \mathscr{F}_1, \ldots$ of families of functions defined as follows:

(1) $\mathscr{F}_0$ consists of all functions of the type (1°) and (2°),

(2) $\mathscr{F}_{i+1}$ consists of all functions of the form

$$g(X_1, \ldots, X_m) = e_j(g_1(X_1, \ldots, X_m), \ldots, g_{k_j}(X_1, \ldots, X_m)),$$

where $g_1, \ldots, g_{k_j} \in \mathscr{F}_i$.

As it is easy to see,

$$\mathrm{IND}[e_1, ..., e_n, \mathscr{F}_0] = \bigcup_{i=1}^{\infty} \mathscr{F}_i. \tag{8}$$

THEOREM 5.1. *Let* $\mathrm{IND}[e_1, ..., e_n, \mathscr{L}_0]$ *be an arbitrary inductive family of functions, where* $e_1, ..., e_n$ *are continuous and let* h *be an arbitrary continuous operation with* $h(\varnothing) = \varnothing$. *If* $\mathscr{L}_0$ *is closed under* h *and if for every* e_i *there exists an* f_i *in* $\mathrm{IND}[e_1, ..., e_n, \mathscr{L}_0]$ *with*

$$h(e_i(X_1, ..., X_m)) = f_i(h(X_1), ..., h(X_m)),$$

then the equational family of languages $\mathrm{EQ}[\mathrm{IND}[e_1, ..., e_n, \mathscr{L}_0]]$ *is closed under the operation* h.

Proof. Let the assumptions of the theorem be satisfied, let

$$F = \mathrm{IND}[e_1, ..., e_n, \mathscr{L}_0]$$

and let $\mathscr{F}_0, \mathscr{F}_1, ...$ be the sequence of families of functions in the sense of (8), where $\mathscr{F}_0 = \mathscr{F}$. We can easily prove by induction the following assertion:

(*) for every $i \geqslant 1$ and every g in $\mathscr{F}_i$ there exists f in $\mathscr{F}$ with $h(g(X_1, ..., X_m)) = f(h(X_1), ..., h(X_m))$ for any $X_1, ..., X_m$ in $\mathscr{L}^1$.

This, by Theorem 4.3, completes the proof of our theorem. Q.E.D.

THEOREM 5.2. *Let* $e_1, ..., e_n$ *be arbitrary continuous functions and let* $\mathscr{L}_0$ *be an arbitrary family of languages.*

If $\mathscr{L}_1 = \mathrm{EQ}[\mathrm{IND}[e_1, ..., e_n, \mathscr{L}_0]]$, *then* $\mathscr{L}_0 \subseteq \mathscr{L}_1$ *and*

$$\mathscr{L}_1 = \mathrm{EQ}[\mathrm{IND}[e_1, ..., e_n, \mathscr{L}_1]].$$

Proof. Let $\mathrm{IND}[e_1, ..., e_n]$ denote the least set of functions that contains $e_1, ..., e_n$ and all projections and that is closed under the operation of composition. The following assertion can easily be proved by (8) and the induction on i. The proof is left to the reader.

(**) For every f in $\mathrm{IND}[e_1, ..., e_n, \mathscr{L}_0]$ there exists g in $\mathrm{IND}[e_1, ..., e_n]$ and $A_1, ..., A_k$ in $\mathscr{L}_0$ with

$$f(X_1, ..., X_m) = g(X_1, ..., X_m, A_1, ..., A_k).$$

Now we can start the proper proof of the theorem. The inclusion $\mathscr{L}_0 \subseteq \mathscr{L}_1$ is obvious since every A in $\mathscr{L}_0$ is the least solution of the equation $X = A$.

Let now

$$\mathrm{IND}_0 = \mathrm{IND}[e_1, \ldots, e_n, \mathscr{L}_0],$$
$$\mathrm{IND}_1 = \mathrm{IND}[e_1, \ldots, e_n, \mathscr{L}_1].$$

Consider the set of equations

$$\mathbf{X} = F(\mathbf{X}),$$

where $F = (f_1, \ldots, f_m)$ with $f_1, \ldots, f_m \in \mathrm{IND}_1$ and let $\mathbf{A} = \| F \|$. We shall show now that $\mathbf{A} \in (\mathscr{L}_1)^m$.

Let $g_1, \ldots, g_m$ be these functions in $\mathrm{IND}[e_1, \ldots, e_n]$ which correspond to $f_1, \ldots, f_m$ in the sense of $(**)$. Without any loss of generality we can assume now the existence of a natural number k and a vector $\mathbf{B}$ in $\mathscr{L}^k$ with the property that $g_1, \ldots, g_m$ are all $m + k$-argumental functions and that

$$f_i(\mathbf{X}) = g_i(\mathbf{X}, \mathbf{B}) \tag{9}$$

for $i = 1, \ldots, m$. This assumption is clearly equivalent with adding to every g_i a corresponding number of "unneccessary" variables. Let now $\mathbf{B} = (B_1, \ldots, B_k)$. By our assumption $B_1, \ldots, B_k \in \mathscr{L}_1$, thus there exist v-functions $F_1, \ldots, F_k$ in IND_0 with

$$[\| F_i \|]_1 = B_i \tag{10}$$

for $i = 1, \ldots, k$. Consider now the following set of equations:

$$\begin{aligned} \mathbf{Z}_1 &= F_1(\mathbf{Z}_1), \\ &\ldots \\ \mathbf{Z}_k &= F_k(\mathbf{Z}_k), \\ Y_1 &= Z_{11}, \\ &\ldots \\ Y_k &= Z_{1k}, \\ \mathbf{X} &= G(\mathbf{X}, \mathbf{Y}), \end{aligned} \tag{11}$$

where $G = (g_1, \ldots, g_m)$, $\mathbf{Y} = (Y_1, \ldots, Y_k)$, $\mathbf{Z}_i = (Z_{1i}, \ldots, Z_{s_i i})$. This is clearly a set of equations in IND_0. Consider now the set of equations:

$$\begin{aligned} \mathbf{Z}_1 &= F_1(\mathbf{Z}_1), \\ &\ldots \\ \mathbf{Z}_k &= F_k(\mathbf{Z}_k), \\ Y_1 &= Z_{11}, \\ &\ldots \\ Y_k &= Z_{1k} \end{aligned}$$

and let $(\mathbf{C}_1, ..., \mathbf{C}_k, \mathbf{D})$ be the least solution of this set. By Theorem 3.1, $\mathbf{D} = ([\|F_1\|]_1, ..., [\|F_k\|]_1)$, thus, by (10) $\mathbf{D} = (B_1, ..., B_k) = \mathbf{B}$. Therefore, once more by Theorem 3.1, the least solution of (11) is of the form

$$(\mathbf{C}_1, ..., \mathbf{C}_k, \mathbf{B}, \mathbf{E}),$$

where $\mathbf{E}$ is the least solution of the set

$$\mathbf{X} = G(\mathbf{X}, \mathbf{B}).$$

Clearly the last set is equivalent to

$$\mathbf{X} = F(\mathbf{X})$$

[see (9)], thus $(\mathbf{C}_1, ..., \mathbf{C}_k, \mathbf{B}, \mathbf{A})$ is the least solution of (11). Therefore $\mathbf{A} \in (L_1)^m$ since (11) is clearly a set in IND_0. Q.E.D.

The above theorem permits strengthening Theorem 5.1 into the following form:

THEOREM 5.3. *Let $e_1, ..., e_n$ be arbitrary continuous functions, let $\mathscr{L}_0$ be an arbitrary family of languages, let $\mathscr{L}_1 = \mathrm{EQ}[\mathrm{IND}[e_1, ..., e_n, \mathscr{L}_0]]$ and let h be an arbitrary continuous operation.*

If $h(\mathscr{L}_0) \subseteq \mathscr{L}_1$ and for every e_i there exists f_i in $\mathrm{IND}[e_1, ..., e_n, \mathscr{L}_1]$ with

$$h(e_i(X_1, ..., X_m)) = f_i(h(X_1), ..., h(X_m)),$$

then the family $\mathscr{L}_1$ is closed under h, i.e., $h(\mathscr{L}_1) \subseteq \mathscr{L}_1$.

Proof. Suppose the assumptions of the theorem are satisfied and let $\mathscr{F}^0 = \mathrm{IND}[e_1, ..., e_n, \mathscr{L}_0]$ and $\mathscr{F}^1 = \mathrm{IND}[e_1, ..., e_n, \mathscr{L}_1]$. In a way analogous to that in the proof of Theorem 5.1 we can easily show the following:

(***) For every f in $\mathscr{F}^0$ there exists an f_1 in $\mathscr{F}^1$ with

$$h(f(X_1, ..., X_m)) = f_1(h(X_1), ..., h(X_m))$$

for any $X_1, ..., X_m$.

Let now $\mathbf{A} \in (\mathscr{L}_1)^m$. By this assumption there exists F in $\mathscr{F}^0$ with $\|F\| = \mathbf{A}$. Hence, by (***), there exists F_1 in $\mathscr{F}^1$ with

$$F \circ h^m = h^m \circ F_1.$$

Let $\mathbf{B} = \|F_1\|$. By Theorem 5.2 $\mathbf{B} \in (\mathscr{L}_1)^m$ and the equality $h^m(\mathbf{A}) = \mathbf{B}$ can be proved analogously as in the proof of Theorem 4.3. Q.E.D.

6. Context-Free Languages

Let $\cup$, $\circ$ and $\mathscr{L}_{\mathrm{F}}$ denote, respectively, the operation of union, the operation of concatenation and the set of all finite languages, the empty languages included, over the alphabet V fixed in Section 2. Let $\mathscr{L}_{\mathrm{CF}}$ denote the class of all context-free languages over V.

By the set of all *standard polynomials* we mean the inductive family $\mathrm{IND}[\cup, \circ, \mathscr{L}_{\mathrm{F}}]$. Clearly, standard polynomials are functions which appear in the so-called Backus–Naur equations in the definition of ALGOL and other similar languages. Therefore, the well-known theorem [see Ginsburg and Rice (1962) and also Ginsburg (1966)] to the effect that ALGOL-like languages and CF languages are the same, has now the following wording:

THEOREM 6.1. $\mathrm{EQ}[\mathrm{IND}[\cup, \circ, \mathscr{L}_{\mathrm{F}}]] = \mathscr{L}_{\mathrm{CF}}$.

According to this theorem CF languages are definable by means of polynomial equations with finite "coefficients." By Theorem 5.2 we can claim now that CF languages can be defined also by polynomial equations with CF coefficients, and by Theorem 4.2 that polynomial equations can be replaced by equations in $\mathrm{RCC}[\mathrm{IND}[\cup, \circ, \mathscr{L}_{\mathrm{CF}}]]$. In other words we have the following:

THEOREM 6.2. $\mathrm{EQ}[\mathrm{IND}[\cup, \circ, \mathscr{L}_{\mathrm{CF}}]] = \mathscr{L}_{\mathrm{CF}}$.

THEOREM 6.3. $\mathrm{EQ}[\mathrm{RCC}[\mathrm{IND}[\cup, \circ, \mathscr{L}_{\mathrm{CF}}]]] = \mathscr{L}_{\mathrm{CF}}$.

Note that the star closure is in $\mathrm{RCC}[\cup, \circ, \mathscr{L}_{\mathrm{CF}}]$ since $Y^* = (\mu X)(YX \cup \{\epsilon\})$. Consequently polynomial equations with star operation define always CF languages. For example the set of equations

$$X = (XY)^* A \cup Y^*,$$
$$Y = XY^*X \cup B,$$

defines two CF languages, provided $A, B \in \mathscr{L}_{\mathrm{CF}}$.

The above theorems together with theorems in Sections 4 and 5 imply immediately some well-known results [see Ginsburg (1966)] concerning closure properties of $\mathscr{L}_{\mathrm{CF}}$:

THEOREM 6.4. *$\mathscr{L}_{\mathrm{CF}}$ is closed under the union, the concatenation and the star closure of languages.*

The proof is by Theorems 6.1 and 4.1.

THEOREM 6.5. *$\mathscr{L}_{\mathrm{CF}}$ is closed under the operation of substitution, i.e., under every substitution s with the property that $s(a) \in \mathscr{L}_{\mathrm{CF}}$ for every a in V.*

Proof. Let s be an arbitrary substitution with $s(a) \in \mathscr{L}_{\mathrm{CF}}$ for every a in V. Since $\mathscr{L}_{\mathrm{CF}}$ is closed under the union and the concatenation, $s(\mathscr{L}_{\mathrm{F}}) \subseteq \mathscr{L}_{\mathrm{CF}}$. Moreover,

$$s(A_1 \cup A_2) = s(A_1) \cup s(A_2)$$

and

$$s(A_1 A_2) = s(A_1)\, s(A_2)$$

for any A_1 and A_2 in $\mathscr{L}^1$. Therefore, by Theorem 5.3 and by the observation that s is continuous, $\mathscr{L}_{\mathrm{CF}}$ is closed under s. Q.E.D.

Also the class $\mathscr{L}_{\mathrm{R}}$ of all regular (finite-state) languages is an equational class of languages. Let SRLP (to be read: standard right-linear polynomials) denote the class of all functions of the form

$$f(X_1, \ldots, X_n) = A_0 \cup A_1 X_1 \cup \cdots \cup A_n X_n, \tag{12}$$

where $A_0, \ldots, A_n$ are finite languages. As it is proved in Blikle (1971),

THEOREM 6.6. $\mathrm{EQ}[\mathrm{SRLP}] = \mathscr{L}_{\mathrm{R}}$.

It can be also proved that finite coefficients in (12) can be replaced by arbitrary regular coefficients and Theorem 6.6 remains true. However SRLP is not an inductive family of functions, thus the theorems given in Section 5 cannot be applied in this case.

A practical application of the theory of equations to the theory of regular languages and finite-state automata is an effective and simple algorithm that associates with every fs automaton the corresponding regular expression. To this effect one needs only write and resolve a simple set of right-linear equations. For details see Blikle (1971).

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